

# On longest non-Hamiltonian Cycles in Digraphs with the Conditions of Bang-Jensen, Gutin and Li

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## Abstract

Let  $D$  be a strong digraph on  $n \geq 4$  vertices. In [2, J. Graph Theory 22 (2) (1996) 181-187], J. Bang-Jensen, G. Gutin and H. Li proved the following theorems: If (\*)  $d(x) + d(y) \geq 2n - 1$  and  $\min\{d(x), d(y)\} \geq n - 1$  for every pair of non-adjacent vertices  $x, y$  with a common in-neighbour or (\*\*)  $\min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq n$  for every pair of non-adjacent vertices  $x, y$  with a common in-neighbour or a common out-neighbour, then  $D$  is hamiltonian. In this paper we show that: (i) if  $D$  satisfies the condition (\*) and the minimum semi-degree of  $D$  at least two or (ii) if  $D$  is not directed cycle and satisfies the condition (\*\*), then either  $D$  contains a cycle of length  $n - 1$  or  $n$  is even and  $D$  is isomorphic to complete bipartite digraph or to complete bipartite digraph minus one arc.

Keywords: Digraphs; cycles; Hamiltonian cycles; longest non-Hamiltonian cycles

## 1. Introduction and Terminology

We shall assume that the reader is familiar with the standard terminology on directed graphs (digraphs) and refer the reader to monograph of Bang-Jensen and Gutin [1] for terminology not discussed here. In this paper we consider finite digraphs without loops and multiple arcs. For a digraph  $D$ , we denote by  $V(D)$  the vertex set of  $D$  and by  $A(D)$  the set of arcs in  $D$ . Often we will write  $D$  instead of  $A(D)$  and  $V(D)$ . The arc of a digraph  $D$  directed from  $x$  to  $y$  is denoted by  $xy$ . For disjoint subsets  $A$  and  $B$  of  $V(D)$  we define  $A(A \rightarrow B)$  as the set  $\{xy \in A(D) / x \in A, y \in B\}$  and  $A(A, B) = A(A \rightarrow B) \cup A(B \rightarrow A)$ . If  $x \in V(D)$  and  $A = \{x\}$  we write  $x$  instead of  $\{x\}$ . If  $A$  and  $B$  are two disjoint subsets of  $V(D)$  such that every vertex of  $A$  dominates every vertex of  $B$ , then we say that  $A$  dominates  $B$ , denoted by  $A \rightarrow B$ . The out-neighbourhood of a vertex  $x$  is the set  $N^+(x) = \{y \in V(D) / xy \in A(D)\}$  and  $N^-(x) = \{y \in V(D) / yx \in A(D)\}$  is the in-neighbourhood of  $x$ . Similarly, if  $A \subseteq V(D)$  then  $N^+(x, A) = \{y \in A / xy \in A(D)\}$  and  $N^-(x, A) = \{y \in A / yx \in A(D)\}$ . We call the vertices in  $N^+(x)$ ,  $N^-(x)$ , the out-neighbours and in-neighbours of  $x$ . The out-degree of  $x$  is  $d^+(x) = |N^+(x)|$  and  $d^-(x) = |N^-(x)|$  is the in-degree of  $x$ . The out-degree and in-degree of  $x$  we call its semi-degrees. Similarly,  $d^+(x, A) = |N^+(x, A)|$  and  $d^-(x, A) = |N^-(x, A)|$ . The degree of the vertex  $x$  in  $D$  defined as  $d(x) = d^+(x) + d^-(x)$  (similarly,  $d(x, A) = d^+(x, A) + d^-(x, A)$ ). The subdigraph of  $D$  induced by a subset  $A$  of  $V(D)$  is denoted by  $\langle A \rangle$ . The path (respectively, the cycle) consisting of the distinct vertices  $x_1, x_2, \dots, x_m$  ( $m \geq 2$ ) and the arcs  $x_i x_{i+1}$ ,  $i \in [1, m - 1]$  (respectively,  $x_i x_{i+1}$ ,  $i \in [1, m - 1]$ , and  $x_m x_1$ ), is denoted  $x_1 x_2 \dots x_m$  (respectively,  $x_1 x_2 \dots x_m x_1$ ). For a cycle  $C_k = x_1 x_2 \dots x_k x_1$ , the subscripts considered modulo  $k$ , i.e.  $x_i = x_s$  for every  $s$  and  $i$  such that  $i \equiv s \pmod{k}$ . If  $P$  is a path containing a subpath from  $x$  to  $y$  we let  $P[x, y]$  denote that subpath. Similarly, if  $C$  is a cycle containing vertices  $x$  and  $y$ ,  $C[x, y]$  denotes the subpath of  $C$  from  $x$  to  $y$ . A digraph  $D$  is strongly connected (or just strong) if there exists a path from  $x$  to  $y$  and a path from  $y$  to  $x$  in  $D$  for every choice of distinct vertices  $x, y$  of  $D$ . A digraph  $D$  is semicomplete if, for every pair of distinct vertices  $x$  and  $y$ , there is at least one arc between them and is locally semicomplete, if  $\langle N^+(x) \rangle$  and  $\langle N^-(x) \rangle$  are both semicomplete for every  $x$  of  $D$ . We will denote the complete bipartite digraph with partite sets of cardinalities  $p, q$  by  $K_{p,q}^*$ . Two

distinct vertices  $x$  and  $y$  are adjacent if  $xy \in A(D)$  or  $yx \in A(D)$  (or both). We denote by  $a(x, y)$  the number of arcs between the vertices  $x$  and  $y$ . In particular,  $a(x, y) = 0$  (respectively,  $a(x, y) \neq 0$ ) means that  $x$  and  $y$  are not adjacent (respectively, are adjacent).

For integers  $a$  and  $b$ ,  $a \leq b$ , let  $[a, b]$  denote the set of all integers which are not less than  $a$  and are not greater than  $b$ . The digraph  $D$  is hamiltonian (is pancyclic, respectively) if it contains a hamiltonian cycle, i.e. a cycle of length  $|V(D)|$  (contains a cycle of length  $m$  for any  $3 \leq m \leq |V(D)|$ ).

Meyniel [12] proved the following theorem: if  $D$  is a strong digraph on  $n \geq 2$  vertices and  $d(x) + d(y) \geq 2n - 1$  for all pairs of non-adjacent vertices in  $D$ , then  $D$  is hamiltonian (for short proofs of Meyniel's theorem see [5, 13]).

Thomassen [15] (for  $n = 2k + 1$ ) and Darbinyan [7] (for  $n = 2k$ ) proved: if  $D$  is a digraph on  $n \geq 5$  vertices with minimum degree at least  $n - 1$  and with minimum semi-degree at least  $n/2 - 1$ , then  $D$  is hamiltonian (unless some extremal cases).

In each above mentioned theorems (as well as, in well known theorems Ghouila-Houri [10], Woodall [16], Manoussakis [11]) imposes a degree condition on all pairs of non-adjacent vertices (on all vertices). Bang-Jensen, Gutin, Li, Guo and Yeo [2, 3] obtained sufficient conditions for hamiltonicity of digraphs in which degree conditions requiring only for some pairs of non-adjacent vertices. Namely, they proved the following theorems (in all three theorems  $D$  is a strong digraph on  $n \geq 2$  vertices).

**Theorem A** [2]. If  $\min\{d(x), d(y)\} \geq n - 1$  and  $d(x) + d(y) \geq 2n - 1$  for every pair of non-adjacent vertices  $x, y$  with a common in-neighbour, then  $D$  is hamiltonian.

**Theorem B** [2]. If  $\min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq n$  for every pair of non-adjacent vertices  $x, y$  with a common out-neighbour or a common in-neighbour, then  $D$  is hamiltonian.

**Theorem C** [3]. If  $\min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq n - 1$  and  $d(x) + d(y) \geq 2n - 1$  for every pair of non-adjacent vertices  $x, y$  with a common out-neighbour or a common in-neighbour, then  $D$  is hamiltonian.

Note that Theorem C generalizes Theorem B. In [9, 14, 6, 8] it was shown that if a strong digraph  $D$  satisfies the condition of the theorem of Ghouila-Houri [10] (Woodall [16], Meyniel [12], Thomassen and Darbinyan [15, 7]), then  $D$  is pancyclic (unless some extremal cases, which are characterized). It is not difficult to check that the digraphs  $K_{n/2, n/2}^*$  and  $K_{n/2, n/2}^* - \{e\}$ , where  $n$  is even and  $e$  is an arc of  $K_{n/2, n/2}^*$ , satisfy the conditions of Theorem A (B, C) and has no cycle of odd length. Moreover, if in Theorems A, B, C the digraph  $D$  has no pair of non-adjacent vertices with a common in-neighbour and a common out-neighbour, then  $D$  is a locally semicomplete digraph, and in [4], Bang-Jensen, Gutin and Volkmann characterize those strong locally semicomplete digraphs which are not pancyclic. For example, the following digraphs  $D(5)$  and  $D(6)$  with 5 and 6 vertices (respectively) are strong locally semicomplete, but has no cycle of length three, where

$$V(D(5)) = \{x_1, \dots, x_4, y\} \text{ and } A(D(5)) = \{x_i x_{i+1} / 1 \leq i \leq 3\} \cup \{x_4 x_1, x_2 y, x_3 y, y x_3, y x_4\};$$

$$V(D(6)) = \{x_1, \dots, x_5, y\} \text{ and } A(D(6)) = \{x_i x_{i+1} / 1 \leq i \leq 4\} \cup \{x_5 x_1, x_1 x_3, x_2 x_4, x_2 y, x_3 y\} \cup \{x_4 y, y x_4, y x_5\}.$$

It is natural to set the following problem:

**Problem.** Characterize those digraphs which satisfy the conditions of Theorem A (B, C), but are not pancyclic.

To investigate that a given digraph  $D$  is pancyclic, in [8, 13, 5, 7] it was proved the existence of cycles of length  $|V(D)| - 1$  or  $|V(D)| - 2$ , and then using the constructions of these cycles it was proved that  $D$  is pancyclic with some exceptions.

In this paper we prove two results which provide some support for the above problem:

(i) if a strong digraph  $D$  satisfies the condition of Theorem A and the minimum semi-degree of  $D$  is at least two; or

(ii) if a strong digraph  $D$  is not a directed cycle and satisfies the condition of Theorem B, then either  $D$  contains a cycle of length  $n - 1$  or  $n$  is even and  $D$  is isomorphic to complete bipartite digraph or to complete bipartite digraph minus one arc.

Our proofs are based on the argument of [2, 3], which was turned based on the ideas used by Bondy, Häggkvist and Thomassen [5, 9, 14].

## 2. Preliminaries

The following well-known simple lemmas is the basis of our results and other theorems on directed cycles and paths in digraphs. It will be used extensively in the proofs of our results.

**Lemma 1** [9]. Let  $D$  be a digraph on  $n \geq 3$  vertices containing a cycle  $C_m$ ,  $m \in [2, n - 1]$ . Let  $x$  be a vertex not contained in this cycle. If  $d(x, C_m) \geq m + 1$ , then  $D$  contains a cycle  $C_k$  for all  $k \in [2, m + 1]$ .

**Lemma 2** [5]. Let  $D$  be a digraph on  $n \geq 3$  vertices containing a path  $P := x_1x_2 \dots x_m$ ,  $m \in [2, n - 1]$  and let  $x$  be a vertex not contained in this path. If one of the following conditions holds:

- (i)  $d(x, P) \geq m + 2$ ;
- (ii)  $d(x, P) \geq m + 1$  and  $xx_1 \notin D$  or  $x_mx_1 \notin D$ ;
- (iii)  $d(x, P) \geq m$ ,  $xx_1 \notin D$  and  $x_mx \notin D$ ;

then there is an  $i \in [1, m - 1]$  such that  $x_ix, xx_{i+1} \in D$ , i.e.  $D$  contains a path  $x_1x_2 \dots x_ixx_{i+1} \dots x_m$  of length  $m$  (we say that  $x$  can be inserted into  $P$  or the arc  $x_ix_{i+1}$  is a partner of  $x$  on  $P$ ).

**Lemma 3** [2]. Let  $P := x_1x_2 \dots x_m$  be a path in  $D$  and let  $x, y$  be vertices of  $V(D) - V(P)$  (possibly  $x = y$ ). If there do not exist consecutive vertices  $x_i, x_{i+1}$  on  $P$  such that  $x_ix, yx_{i+1}$  are arcs of  $D$ , then  $d^-(x, P) + d^+(y, P) \leq m + \epsilon$ , where  $\epsilon = 1$  if  $x_mx \in D$  and 0, otherwise.

## 3. Main results

Let  $C$  be a cycle in digraph  $D$ . For the cycle  $C$ , a  $C$ -bypass is an  $(x, y)$ -path  $P$  of length at least two with both end-vertices  $x$  and  $y$  on  $C$  and no other vertices on  $C$ . The length of the path  $C[x, y]$  is the gap of  $P$  with respect to  $C$ .

In the proof of Theorem 1, if  $\{x, y\}$  is a pair of non-adjacent vertices with a common in-neighbour, then we say that  $\{x, y\}$  is a good pair.

In the proofs of Theorems 1 and 2 we use (in the main) the notations which are used in the proofs of Theorems A and B (see [1], Theorems 5.6.1 and 5.6.5, pages 248-250).

**Theorem 1.** Let  $D$  be a strong digraph on  $n$  vertices with minimum semi-degree at least two. Suppose that

$$d(x) + d(y) \geq 2n - 1 \quad \text{and} \quad \min\{d(x), d(y)\} \geq n - 1 \quad (*)$$

for every pair of non-adjacent vertices  $x$  and  $y$  with a common in-neighbour. Then either  $D$  contains a cycle of length  $n - 1$  or  $n$  is even and  $D$  is isomorphic to complete bipartite digraph  $K_{n/2, n/2}^*$  or  $K_{n/2, n/2}^* - \{e\}$ , where  $e$  is an arc of  $K_{n/2, n/2}^*$ .

**Proof.** If  $n \leq 4$ , the theorem is easily verified. Suppose that  $n \geq 5$  and  $D$  contain no cycle of length  $n - 1$  and let  $C := x_1x_2 \dots x_mx_1$  be a longest non-hamiltonian cycle in  $D$ . Then  $3 \leq m \leq n - 2$  and let  $R := V(D) - V(C)$ . Observe that if  $y \notin V(C)$ , then  $y$  has no partner on  $C$ . We shall use this often without explicit reference.

We first prove the following claim:

**Claim 1.** Let  $|R| \geq 3$  and  $x_1yx_{\alpha+1}$  be a  $C$ -bypass of length two. If  $A(y, C[x_2, x_\alpha]) = \emptyset$ , then  $\alpha \geq 4$ .

**Proof.** Suppose that  $\alpha \leq 3$ . Observe that  $\{x_2, y\}$  is a good pair. Since  $C$  is a longest non-hamiltonian cycle in  $D$  and  $|R| \geq 3$ , it is not difficult to see that

$$d^+(y, R) + d^-(x_i, R) \geq n - m - 1 \quad \text{and} \quad d^-(y, R) + d^+(x_i, R) \leq n - m - 1 \quad (1)$$

for every  $i \in \{2, \alpha\}$  and by Lemma 2(i),

$$d(y, C) = d(y, C[x_{\alpha+1}, x_1]) \leq m - \alpha + 2. \quad (2)$$

If  $\alpha = 2$ , then the vertex  $x_2$  also cannot be inserted into  $C[x_{\alpha+1}, x_1]$  and hence by Lemma 2(i),  $d(x_2) \leq m$ . This together with (1) and (2) implies that  $d(y) + d(x_2) \leq 2n - 2$ , which is a contradiction since  $\{y, x_2\}$  is a good pair. So we can assume that  $\alpha = 3$ . Now for  $i \in \{2, 3\}$  using (1) and (2) we obtain that

$$d(y) + d(x_i) \leq 2n - m - 3 + d(x_i, C). \quad (3)$$

It is clear that if  $x_1x_i \in D$  ( $i \in \{2, 3\}$ ), then  $\{y, x_i\}$  is a good pair and by (\*) and (3) we obtain that  $d(x_i, C) \geq m + 2$ . Therefore, by Lemma 2(i),  $x_2$  has a partner on  $C[x_4, x_1]$ . This means that  $x_lx_{l+1} \in D$  for some  $l \in [4, m]$  and  $x_3$  has no partner on  $C[x_4, x_1]$  (otherwise, we obtain a non-hamiltonian cycle longer than  $C$ ). Now from above observation it follows that  $x_1x_3 \notin D$ . Observe that  $x_3x_{l+1} \notin D$ . If  $a(x_{l+1}, x_3) = 0$ , then  $\{x_3, x_{l+1}\}$  is a good pair and  $d(x_3) \geq n - 1$  by (\*). It follows from (3) and  $d(y) \geq n - 1$  that  $d(x_3, C[x_4, x_1]) \geq m - 1$  and hence by Lemma 2(ii),  $x_1x_3 \in D$ , which is a contradiction. So we can assume that  $x_{l+1}x_3 \in D$ ,  $x_1 \neq x_{l+1}$ . Considering the pair  $\{x_3, x_{l+2}\}$ , we conclude analogously that  $x_{l+2}x_3 \in D$ . Continuing this process, we finally conclude that  $x_1x_3 \in D$ , contradicting the conclusion above that this arc does not exist. Claim 1 is proved.  $\square$

In [2] (see [1], page 248), is proved that  $D$  contains a  $C$ -bypass  $P := u_1u_2 \dots u_s$  ( $s \geq 3$ ). W.l.o.g., let  $u_1 := x_1$ ,  $u_s := x_{\gamma+1}$ ,  $0 < \gamma < m$ . Suppose also that the gap  $\gamma$  of  $P$  is minimum among the gaps of all  $C$ -bypasses, i.e.

$$A(\{u_2, \dots, u_{s-1}\}, C[x_2, x_\gamma]) = \emptyset. \quad (4)$$

Let  $C' := C[x_2, x_\gamma]$  and  $C'' := C[x_{\gamma+1}, x_1]$ . Note that if  $\gamma \geq 2$ , then  $\{u_2, x_2\}$  is a good pair and by (\*)

$$\min\{d(u_2), d(x_2)\} \geq n - 1 \quad \text{and} \quad d(u_2) + d(x_2) \geq 2n - 1, \quad (5)$$

We first show that  $m \geq n - 2$ , i.e.  $|R| \leq 2$ . For this it suffices to consider the following four cases.

**Case 1.**  $|R| \geq 3$  and  $|R - P[u_2, u_{s-1}]| \geq 2$ . The discussion of this case exactly is as same as the proof of Theorem 5.6.1 (see [1], page 249).

**Case 2.**  $|R| \geq 3$  and  $|R - P[u_2, u_{s-1}]| = 1$ . Then  $\gamma \geq 3$  since  $|P[u_2, u_{s-1}]| \geq 2$ . Let  $R - P[u_2, u_{s-1}] = \{y\}$ . We can assume that  $d^+(u_2, \{u_4, \dots, u_{s-1}\}) = 0$  and  $u_2x_{\gamma+1} \notin D$  (otherwise, we have Case 1). Therefore  $d(u_2, R - \{y\}) \leq |R| - 1 = n - m - 1$ ,  $d(u_2, R) \leq n - m + 1$ , and by Lemma 2(ii),  $d(u_2, C'') \leq |C''| = m - \gamma + 1$ . This together with (4) and (5) implies that

$$n - 1 \leq d(u_2) = d(u_2, C'') + d(u_2, R - \{y\}) + a(u_2, y) \leq n - \gamma + a(u_2, y).$$

From this it is easy to see that  $\gamma = 3$ ,  $a(u_2, y) = 2$ ,  $d(u_2) = n - 1$ ,  $|R| = 3$  since  $\gamma \geq |R|$ ,  $d(u_2, C'') = n - 5$  and  $a(u_2, u_3) = 2$  ( $s = 4$ ). Then by (5),  $d(x_2) \geq n$ . Now since  $d(x_2, C') \leq 2$  and  $d(x_2, R) \leq 1$ , it follows that  $d(x_2, C'') \geq n - 3 \geq |C''| + 2$ . Therefore, by Lemma 2(i),  $x_2$  has a partner on  $C''$ , i.e.  $x_i x_2, x_2 x_{i+1} \in D$ , where  $i \in [4, m]$ . Thus the non-hamiltonian cycle  $x_1 u_2 u_3 x_4 \dots x_i x_2 x_{i+1} \dots x_m x_1$  has length  $n - 2$ , which is a contradiction.

**Case 3.**  $|R| \geq 3$ ,  $R = P[u_2, u_{s-1}]$  and  $\gamma \neq 1$ . Then  $\gamma \geq |R| + 1$ , and  $d(u_2, C'') \leq |C''|$  since  $u_2 x_{\gamma+1} \notin D$ . We can assume that  $d(u_2, R) \leq n - m$  (otherwise, we have Case 1 or 2). Therefore

$$n - 1 \leq d(u_2) = d(u_2, C'') + d(u_2, R) \leq |C''| + n - m \leq n - 3$$

since  $|C''| = m - \gamma + 1$ , which is a contradiction.

**Case 4.**  $|R| \geq 3$ ,  $R = P[u_2, u_{s-1}]$  and  $\gamma = 1$  (i.e.  $x_{\gamma+1} = x_2$ ). We can assume that if  $2 \leq i < j \leq s - 1$ , then  $u_i u_j \in D$  if and only if  $j = i + 1$  (otherwise, we have one of the cases 1-3). Hence  $d(u_2, R) \leq n - m$ . Observe that

$$d^-(u_2, \{x_{m-1}, x_m\}) = 0 \quad \text{and} \quad u_{s-1} x_3 \notin D. \quad (6)$$

**Subcase 4.1.**  $x_l u_2 \in D$  for some  $x_l \neq x_1$ . Then by (6) there is a vertex  $x_k$  with  $2 \leq k \leq m - 2$  such that  $x_k u_2 \in D$  and  $a(u_2, x_{k+1}) = 0$ . Note that  $x_k \rightarrow \{u_2, x_{k+1}\}$  and  $\{u_2, x_{k+1}\}$  is a good pair. By (\*), we have

$$\min\{d(u_2), d(x_{k+1})\} \geq n - 1 \quad \text{and} \quad d(u_2) + d(x_{k+1}) \geq 2n - 1 \quad (7)$$

Assume that  $k$  is maximal with these properties. If  $d^+(u_2, C[x_{k+2}, x_1]) = 0$ , then  $A(u_2, C[x_{k+1}, x_m]) = \emptyset$  by the maximality of the  $k$ , and by Lemma 2(ii),  $d(u_2, C) = d(u_2, C[x_1, x_k]) \leq k$  since  $u_2 x_1 \notin D$ . Therefore  $d(u_2) \leq k + n - m \leq n - 2$ , which is a contradiction. So we can assume that  $d^+(u_2, C[x_{k+2}, x_1]) \neq 0$ . Then there is an integer  $\alpha \geq 1$ ,  $k + \alpha \leq m$  such that  $u_2 x_{k+1+\alpha} \in D$  and  $A(u_2, C[x_{k+1}, x_{k+\alpha}]) = \emptyset$ . By Claim 1 we have  $\alpha \geq 3$ , and by Lemma 2(i),  $d(u_2, C) \leq m - \alpha + 1$ . This along with  $d(u_2, R) \leq n - m$  and (7) implies that

$$n - 1 \leq d(u_2) = d(u_2, R) + d(u_2, C) \leq n - \alpha + 1 \leq n - 2,$$

which is a contradiction.

**Subcase 4.2.**  $d^-(u_2, C[x_2, x_m]) = 0$ . Then  $x_1 \rightarrow \{u_2, x_2\}$ ,  $a(x_2, u_2) = 0$  and  $\{y, x_2\}$  is a good pair. Therefore, by (\*)

$$\min\{d(u_2), d(x_2)\} \geq n - 1 \quad \text{and} \quad d(u_2) + d(x_2) \geq 2n - 1 \quad (8)$$

Note that  $n - 1 \leq d(u_2) \leq n$ . If  $d(u_2) = n$ , then it is not difficult to see that  $d(u_2, C) = m$ ,  $d(u_2, R) = n - m$ ,  $u_2 \rightarrow C[x_3, x_1]$ ,  $R - \{u_2\} \rightarrow y$  and  $d^+(x_2, R) = d^-(x_2, R - \{u_{s-1}\}) = 0$ . Therefore, since  $x_2$  cannot be inserted into  $C[x_3, x_1]$ , we have  $d(x_2) = d(x_2, R) + d(x_2, C) \leq m + 1 \leq n - 2$ , which contradicts (8). So we can assume that  $d(u_2) = n - 1$  and by (8),  $d(x_2) \geq n$ .

If  $u u_2 \notin D$  for some vertex  $u \in R - \{u_2\}$ , then it is easy to see that  $R - \{u, u_2\} \rightarrow u_2$  and  $u_2 \rightarrow C[x_3, x_1]$ . Now we have

$$d^-(x_2, R - \{u_{s-1}\}) = d^+(x_2, R - \{u\}) = 0$$

and hence,  $d(x_2, R) \leq 2$ . Therefore  $n \leq d(x_2) = d(x_2, C) + d(x_2, R) \leq m + 2 \leq n - 1$ , which is a contradiction. Suppose that this is not the case, i.e.  $R - \{u_2\} \rightarrow u_2$ . Then  $d^-(x_2, R - \{u_{s-1}\}) = 0$ . If  $y x_3 \in D$ , then it is easy to see that  $d(x_2, C) \leq m$  and  $d^+(x_2, R) = 0$ . Therefore  $d(x_2) \leq m + 1 \leq n - 2$ , a contradiction. So we can assume that  $u_2 x_3 \notin D$ . Then  $u_2 \rightarrow C[x_4, x_1]$ ,  $d(x_2, R) \leq 1$ ,  $A(x_3, \{u_2, u_{s-1}\}) = \emptyset$  and  $d^-(u_{s-1}, C[x_1, x_m]) = 0$  since  $u_{s-1} u_2 \in D$ . Therefore  $d^-(u_{s-1}) = 1$  since  $d^-(u_{s-1}, R) = 0$ , which contradicts that  $d^-(u_{s-1}) \geq 2$ .

Thus if  $|R| \geq 3$ , then in all possible cases we have obtained a contradiction. Therefore we have proved that  $m = n - 2$ .  $\square$

Let  $R = \{y, z\}$ . We first prove the following Claims 2-7.

**Claim 2.** If  $x_i y, y x_{i+2} \in D$ ,  $a(y, x_{i+1}) = 0$  and  $d(x_{i+1}) = n$ , then  $a(z, x_{i+1}) = 2$ .

**Proof.** The proof of the claim immediately follows from the maximality of  $C$  and Lemma 1.  $\square$

**Claim 3.** If  $x_1 y \in D$  and  $a(y, x_2) = 0$  (i.e.,  $\{x_2, y\}$  is a good pair). Then  $a(y, x_3) \neq 0$ .

**Proof.** Suppose that the claim is not true, i.e.  $a(y, x_3) = 0$ . By (\*),

$$\min\{d(y), d(x_2)\} \geq n - 1 \quad \text{and} \quad d(y) + d(x_2) \geq 2n - 1 \quad (9)$$

Since  $y$  cannot be inserted into  $C[x_4, x_1]$ , by Lemma 2(i) we have  $d(y, C[x_4, x_1]) \leq n - 3$  (we can assume that  $n \geq 6$ ). Therefore

$$n - 1 \leq d(y) = a(y, z) + d(y, C[x_4, x_1]) \leq n - 1.$$

This implies that  $a(y, z) = 2$ ,  $d(y, C[x_4, x_1]) = n - 3$  and  $d(y) = n - 1$ . Therefore  $d(x_2) \geq n$  (by (9)) and  $y x_4 \in D$  (by Lemma 2(ii)). Since  $C$  is a longest non-hamiltonian cycle in  $D$  and  $a(y, z) = 2$ ,  $y x_4 \in D$ , it follows that  $x_2 z \notin D$ ,  $z x_3 \notin D$  and  $d(x_2, C[x_4, x_1]) \geq n - 3$ .

Now we consider the following two possible cases.

**Case 1.**  $a(x_2, z) = 0$ . Then  $d(x_2, C[x_4, x_1]) \geq n - 2$  and by Lemma 2(i)  $x_2$  has a partner on  $C[x_4, x_1]$ ,  $x_1 z \notin D$  and  $z x_4 \notin D$ . By Lemma 2(iii),  $d(z, C[x_4, x_1]) \leq n - 5$ . Therefore  $d(z) \leq n - 2$ , since  $d(z, \{y, x_3\}) \leq 3$ . This means that  $z$  does not form a good pair with any vertex of  $D$ . Thus we have  $d^-(z, C[x_3, x_1]) = 0$  since  $x_1 z \notin D$ . Therefore  $d^-(z) = 1$ , which is a contradiction.

**Case 2.**  $z x_2 \in D$ . Then  $x_2 x_4 \notin D$  and  $x_m y \notin D$  (otherwise,  $C$  is not longest non-hamiltonian cycle in  $D$ ). Since  $d(x_2, C[x_4, x_1]) \geq n - 3$  and  $x_2 x_4 \notin D$ , using Lemma 2(ii) we obtain,  $x_2$  has a partner on  $C[x_4, x_1]$ , i.e.  $x_i x_2, x_2 x_{i+1} \in D$  for some  $i \in [4, m]$ . Observe that  $z x_4 \notin D$  and  $x_1 z \notin D$ , and by Lemma 2(iii),  $d(z, C[x_4, x_1]) \leq n - 5$ . Therefore  $d(z) \leq n - 1$  since  $d(z, \{x_2, x_3\}) \leq 2$ .

If  $a(x_3, z) = 0$ , then  $d(z) \leq n - 2$  and  $z$  does not form a good pair with any vertex of  $D$ , which is not possible (since  $y x_4 \in D$ ,  $z x_4 \notin D$  and  $x_1 z \notin D$ ). Therefore  $a(x_3, z) \neq 0$ , i.e.  $x_3 z \in D$ . It is easy to see that  $y x_5 \notin D$  and  $x_m y \notin D$ . From this we obtain that  $m \geq 5$ . Now using Lemma 2(iii) we obtain,  $d(y, C[x_5, x_m]) \leq n - 7$  and  $y x_1, x_4 y \in D$ . Since  $z$  has no partner on  $C$  and  $z x_2, x_3 z \in D$ , there is a vertex  $x_l$  with  $l \in [4, m + 1]$  such that  $x_{l-1} z \in D$  and  $a(z, x_l) = 0$  (i.e.,  $z$  forms a good pair with  $x_l$ ). By (\*) and  $d(z) \leq n - 1$  we have that  $d(z) = n - 1$ . It follows that  $z x_{l+1} \in D$  (by Lemma 2(i)). Let  $l$  is minimal with these properties. Since  $d(z) = n - 1$ , then  $d(x_l) \geq n$  (by (\*)),  $d(x_l, C[x_{l+1}, x_{l-1}]) \leq n - 2$  (by Lemma 2(i)) and  $a(y, x_l) = 2$  (Claim 2). Now we have, if  $x_{l-2} z \in D$ , then  $x_{l-2} z y x_l x_{l+1} \dots x_{l-2}$  is a cycle of length  $n - 1$ . Therefore  $x_{l-2} z \notin D$ , i.e.  $l = 4$ . This together with  $x_2 x_4 \notin D$  and  $d(x_4) \geq n$  implies that  $x_4 x_3 \in D$  and  $x_1 y x_4 x_3 z x_5 \dots x_m x_1$  is a cycle of length  $n - 1$ , a contradiction. Claim 3 is proved.  $\square$

**Claim 4.**  $d^-(y, \{x_i, x_{i+1}\}) \leq 1$  for all  $i \in [1, m]$ .

**Proof.** Suppose, on the contrary, that (say)  $\{x_m, x_1\} \rightarrow y$ . W.l.o.g. we assume that  $a(y, x_2) = 0$  (otherwise,  $C \rightarrow y$ ,  $d^+(y, C) = 0$  and hence,  $d^+(y) \leq 1$ , a contradiction). This means that  $\{y, x_2\}$  is a good pair. Therefore for  $y$  and  $x_2$  the condition (\*) holds, i.e.

$$\min\{d(y), d(x_2)\} \geq n - 1 \quad \text{and} \quad d(y) + d(x_2) \geq 2n - 1.$$

By Claim 3, we have that  $a(y, x_3) \neq 0$ . Since the minimum semi-degree of  $D$  is at least two, it is not difficult to see that  $m \geq 4$ . Consider the following three possible cases.

**Case 1.**  $yx_3 \in D$  and  $yz \in D$ . Then  $d^+(z, \{x_2, x_3\}) = 0$  and  $d(x_2, C[x_3, x_1]) \leq n - 2$  (by Lemma 2(i)). Therefore  $d(x_2) = n - 1$  and  $d(y) = n$  (by (\*)), and  $zy, x_2z \in D, x_1z \notin D, x_mx_2 \notin D$ .

First assume that  $a(z, x_3) = 0$ . Because of  $x_2 \rightarrow \{z, x_3\}$ ,  $\{z, x_3\}$  is a good pair and by (\*),  $d(z), d(x_3) \geq n - 1$ . Since  $x_1z \notin D$ , using Lemma 2(ii), we obtain that  $zx_4 \in D$  and  $d(z) = n - 1$  (otherwise,  $d(z) = d(z, \{y, x_2\}) + d(z, C[x_4, x_1]) \leq n - 2$ , a contradiction). Hence  $d(x_3) \geq n$  by (\*), and  $x_3y \in D$  by Claim 2. It is easy to see that  $x_1x_3 \notin D$ , and by Lemma 2(ii),  $x_3x_2 \in D$  since  $d(x_3) \geq n$  and  $x_3$  cannot be inserted into  $C[x_4, x_1]$ . Therefore  $x_myx_3x_2zx_4 \dots x_m$  is a cycle of length  $n - 1$ , a contradiction.

Second assume that  $x_3z \in D$ . Then since  $\{x_2, x_3\} \rightarrow z$  and  $x_1z \notin D$  there is an integer  $k \in [3, m]$  such that  $\{x_{k-1}, x_k\} \rightarrow z$  and  $a(z, x_{k+1}) = 0$ , i.e.  $\{x_{k+1}, z\}$  is a good pair. Since  $zx_2 \notin D$  and  $x_1z \notin D$  using Lemma 2(ii) we obtain,

$$d(z) = d(z, C[x_2, x_k]) + d(z, C[x_{k+2}, x_1]) + a(z, y) \leq n - 1.$$

This together with (\*) and  $d^+(z) \geq 2$  implies that  $d(x_{k+1}) \geq n$  and  $zx_{k+2} \in D$ . Therefore  $a(y, x_{k+1}) = 2$  (Claim 2) and  $x_1x_2 \dots x_{k-1}zyx_{k+1} \dots x_mx_1$  is a cycle of length  $n - 1$ , a contradiction.

**Case 2.**  $yx_3 \in D$  and  $yz \notin D$ . From  $d(y) \geq n - 1$  and  $d(y, C[x_3, x_1]) \leq n - 2$  it follows that  $zy \in D$  and  $d(y) = n - 1$ . Then  $d(x_2) \geq n$  (by (\*)) and  $a(x_2, z) = 2$  (Claim 2). Hence  $yx_4 \notin D$  and by Lemma 2(ii),  $d(y, C[x_4, x_1]) \leq n - 4$ . This along with  $yz \notin D$  and  $d(y) = n - 1$  implies that  $x_3y \in D$ . Observe that  $x_mx_2 \notin D$  and by Lemma 2(ii),  $d(x_2, C[x_3, x_m]) \leq n - 4$  since  $x_2$  cannot be inserted into  $C[x_3, x_m]$ . Therefore  $x_2x_1 \in D$ .

**Subcase 2.1.**  $a(z, x_3) = 0$ . Then  $\{z, x_3\}$  is a good pair. This together with (\*),  $yz \notin D$  and Lemma 2(i) implies that  $d(z) = n - 1$ ,  $zx_4 \in D$  and  $d(x_3) \geq n$ . It is clear that  $x_3x_2 \notin D$  (otherwise,  $C_{n-1} := x_myx_3x_2zx_4 \dots x_m$ ). Now again using Lemma 2(i), we obtain that  $x_1x_3 \in D$  since  $d(x_3, C[x_4, x_1]) \geq n - 3$ . Observe that  $x_mz \notin D$  (otherwise,  $C_{n-1} := x_mzx_2x_1x_3 \dots x_m$ ) and

$$n - 1 = d(z) = d(z, C[x_4, x_m]) + d(z, \{y, x_1, x_2, x_3\}) \leq n - 1.$$

From this it follows that  $zx_1 \in D$ . Let  $m \geq 5$ . Then  $x_3x_5 \notin D$  (otherwise,  $C_{n-1} := x_1x_2zyx_3x_5 \dots x_mx_1$ ) and by Lemma 2(ii),  $x_4x_3 \in D$  since  $d(x_3) = n$ . Thus we have a cycle  $C_{n-2} := x_1x_2zx_4 \dots x_mx_1$  which does not contain the vertices  $y, x_3$  and  $\{x_1, x_2\} \rightarrow x_3$ ,  $a(z, x_3) = 0$  and  $x_3y, x_3x_4 \in D$ . Therefore for this cycle  $C_{n-2}$  Case 1 holds.

Let now  $m = 4$ , i.e.  $n = 6$ . Then because of  $d^-(z, \{y, x_1, x_3, x_m\}) = 0$  we have  $d^-(z) = 1$ , which is a contradiction.

**Subcase 2.2.**  $a(z, x_3) \neq 0$ . Then  $x_3z \in D$  since  $a(x_2, z) = 2$ . Note that  $x_{m-1}z \notin D$  (otherwise,  $C_{n-1} := x_{m-1}zx_2x_1yx_3 \dots x_{m-1}$ ). Then  $m \geq 5$  and it is easy to see that there is an integer  $k \in [3, m - 2]$  so that  $\{x_{k-1}, x_k\} \rightarrow z$ ,  $a(z, x_{k+1}) = 0$ . Then  $\{z, x_{k+1}\}$  is a good pair since  $x_k \rightarrow \{z, x_{k+1}\}$ . From this by (\*),  $d(z) \geq n - 1$ , and by Lemma 2 (i),  $zx_{k+2} \in D$ . Therefore for the vertex  $z$  we have considered Case 1.

**Case 3.**  $yx_3 \notin D$ . Then  $x_3y \in D$  by Claim 3. Using Lemma 2(ii) and (\*), it is not difficult to see that

$$d(y, C) = n - 3, a(y, z) = 2, d(y) = n - 1, d(x_2) \geq n \text{ and } d^+(z, \{x_2, x_3, x_5\}) = 0.$$

If  $x_4y \in D$ , then  $m \geq 6$  and from  $d^+(y) \geq 2$  and  $d(y) = n - 1$  it follows that  $\{x_{k-1}, x_k\} \rightarrow y \rightarrow \{x_{k+2}\}$  and  $a(y, x_{k+1}) = 0$  for some  $k \in [4, m - 2]$ , i.e. we have the considered Case 1.

So we can assume that  $x_4y \notin D$ . Then  $m \geq 5$ ,  $a(y, x_4) = 0$  and  $\{y, x_4\}$  is a good pair since  $x_3 \rightarrow \{x_4, y\}$ . Therefore by (\*),  $d(x_2)$  and  $d(x_4) \geq n$  since  $d(y) = n - 1$ . From  $d(y, \{x_2, x_3, x_4, z\}) = 3$  it follows that  $d(y, C[x_5, x_1]) = n - 4$ . Hence  $yx_5 \in D$  (by Lemma 2(ii)) and  $a(x_4, z) = 2$  (Claim 2). Now it is easy to see that  $x_2x_4 \notin D$ . Again using Lemma 2(ii), we obtain that  $d(x_2, C[x_4, x_1]) \leq n - 4$  and  $d(x_2) \leq n - 1$  since  $zx_2 \notin D$ , contradicting the conclusion above that  $d(x_2) \geq n$ . Claim 4 is proved.  $\square$

**Claim 5.** If  $x_iy \in D$  and  $a(y, x_{i+1}) = 0$ , then  $d^+(y, \{x_{i+2}, x_{i+3}\}) \leq 1$  for all  $i \in [1, m]$ .

**Proof.** Suppose that the claim is not true. W.l.o.g we can assume that  $x_1y \in D$ ,  $a(y, x_2) = 0$  and  $y \rightarrow \{x_3, x_4\}$ . Note that  $\{y, x_2\}$  is a good pair.

**Case 1.**  $zy \notin D$ . Then from  $d(y, C) \leq n - 2$  and (\*) it follows that  $yz \in D$ ,  $d(y) = n - 1$  and  $d(x_2) \geq n$ . Observe that  $a(x_2, z) = 2$  (Claim 2) and  $a(z, x_3) = 0$  (Claim 4). Therefore  $\{z, x_3\}$  also is a good pair and for its (\*) holds. Using Lemma 2(i) and (\*), it is not difficult to see that  $d(z) = n - 1$ ,  $d(x_3) \geq n$  and  $zx_4 \notin D$  since  $zy \notin D$ . Now by Claim 2,  $a(x_3, y) = 2$ , which is a contradiction.

**Case 2.**  $zy \in D$ . Then  $d^-(z, \{x_1, x_2\}) = 0$ ,  $d(x_2, C) = n - 2$ ,  $zx_2 \in D$ ,  $d(y) = n$ ,  $yz \in D$  and  $zx_3 \notin D$  (i.e., either  $x_3z \in D$  or  $a(z, x_3) = 0$ ) by Lemma 2(i) and (\*).

**Subcase 2.1.**  $a(z, x_3) \neq 0$ . Then it is easy to see that  $x_3z \in D$ ,  $m \geq 4$  and  $yx_5 \notin D$ ,  $d(y, C[x_5, x_1]) = n - 5$ ,  $x_4y \in D$  since  $d(y) = n$ . By Claim 4,  $x_4z \notin D$ . Therefore  $a(z, x_4) = 0$  and  $\{z, x_4\}$  is a good pair and hence,  $d(z), d(x_4) \geq n - 1$ . Using Lemma 2(ii) and (\*) we obtain that  $zx_5 \in D$ ,  $d(z) = n - 1$ ,  $d(x_4) = n$  since  $x_2z \notin D$ . If  $x_2x_4 \in D$ , then  $C_{n-1} := x_2x_4yzx_5 \dots x_mx_1x_2$ , and if  $x_2x_4 \notin D$ , we obtain that  $x_4x_3 \in D$  (since  $x_4$  cannot be inserted into  $C[x_5, x_3]$  and  $d(x_4) = n$ ) and  $C_{n-1} := x_1yx_4x_3zx_5 \dots x_mx_1$ , which is a contradiction.

**Subcase 2.2.**  $a(z, x_3) = 0$ . Then  $\{z, x_3\}$  is a good pair since  $y \rightarrow \{z, x_3\}$ , and  $m \geq 4$  since  $d^-(z) \geq 2$ . Therefore by (\*),  $x_2z \notin D$  and Lemma 2(ii), we obtain that  $zx_4 \in D$ ,  $d(z) = n - 1$  and  $d(x_3) \geq n$ . Since the vertex  $x_3$  has no partner on the cycle  $C_{n-2} := x_1yzx_4 \dots x_mx_1$  and  $d(x_3) \geq n$ , using Lemma 2(i) we obtain that  $x_1x_3 \in D$ . Now for this cycle  $C_{n-2}$  we have  $\{x_1, y\} \rightarrow x_3$ , which contradicts Claim 4. Claim 5 is proved.  $\square$

**Claim 6.** If  $a(y, z) = 1$ , then  $n$  is even and  $D \equiv K_{n/2, n/2}^* - \{e\}$ , where  $e$  is an arc of  $K_{n/2, n/2}^*$ .

**Proof.** Since  $D$  contain no cycle of length  $n - 1$ , using Lemma 1 we obtain that  $d(y), d(z) \leq n - 1$ . W.l.o.g. assume that  $yz, x_my \in D$ ,  $a(y, x_1) = 0$  (if  $C \rightarrow y$ , then  $D$  contains a cycle of length  $n - 1$  since  $d^+(z, C) \neq 0$ ). Then  $\{x_1, y\}$  is a good pair and hence,  $d(y) = n - 1$ ,  $yx_2 \in D$  and  $d(x_1) \geq n$  by Lemma 2(i) and (\*). We have  $a(x_1, z) = 2$  (Claim 2), and  $d(x_1, C[x_2, x_m]) = n - 2$  by Lemma 2(i). Therefore  $d^-(z, \{x_m, x_2\}) = 0$  (Claim 4) and  $a(z, x_2) = 0$ , i.e.  $\{z, x_2\}$  is a good pair,  $d(x_2) \geq n$  and by Lemma 2(ii),  $d(z, C[x_3, x_m]) = n - 4$ ,  $zx_3 \in D$  (since  $x_mz \notin D$ ). From this  $a(x_2, y) = 2$  (Claim 2),  $zx_4 \notin D$ ,  $a(y, x_3) = 0$  (Claim 4),  $d(x_3) \geq n$  (by (\*)) and  $d(y, C[x_4, x_m]) = n - 4$ . Therefore  $yx_4 \in D$ ,  $x_3z \in D$  (Claim 2),  $a(z, x_4) = 0$  (Claim 4),  $d(x_4) \geq n$  (by (\*)) and  $zx_5 \in D$ . Continuing this process, we finally conclude that  $n$  is even,  $d(x_i) = n := 2k$ ,

$$y \rightarrow \{x_2, x_4, \dots, x_{2k-2}\} \rightarrow y, \quad z \rightarrow \{x_1, x_3, \dots, x_{2k-3}\} \rightarrow z$$

and

$$A(y, \{x_1, x_3, \dots, x_{2k-3}\}) = A(z, \{x_2, x_4, \dots, x_{2k-2}\}) = \emptyset.$$

Now we prove that

$$A(\langle \{x_1, x_3, \dots, x_{2k-3}\} \rangle) = A(\langle \{x_2, x_4, \dots, x_{2k-2}\} \rangle) = \emptyset.$$

Suppose this is not the case. Let  $x_ix_j \in D$ , where  $i, j \in \{1, 3, \dots, 2k - 3\}$ . Then

$$a(x_i, z) = a(x_j, z) = a(x_{i-1}, y) = a(x_{i+1}, y) = a(x_{j-1}, y) = a(x_{j+1}, y) = 2.$$



If  $|C[x_i, x_j]| = 3$ , then  $C_{n-1} := x_i x_j \dots x_{i-1} y z x_i$ ; if  $|C[x_i, x_j]| \geq 5$ , then  $C_{n-1} := x_i x_j \dots x_{i-1} y x_{i+1} \dots x_{j-2} z x_i$ . Let now  $x_i x_j \in D$ , where  $i, j \in \{2, 4, \dots, 2k-2\}$ . Then

$$a(x_i, y) = a(x_j, y) = a(x_{i-1}, z) = a(x_{i+1}, z) = a(x_{j-1}, z) = a(x_{j+1}, z) = 2.$$

If  $|C[x_i, x_j]| = 3$ , then  $C_{n-1} := x_i x_j \dots x_{i-2} y z x_{i-1} x_i$ ; if  $|C[x_i, x_j]| \geq 5$ , then  $m \geq 6$  and  $C_{n-1} := x_i x_j \dots x_{i-1} z x_{i+1} \dots x_{j-2} y x_i$ . In all possible cases we have that  $D$  contains a cycle of length  $n-1$ , which is a contradiction. Therefore

$$A(\langle \{y, x_1, x_3, \dots, x_{2k-3}\} \rangle) = A(\langle \{z, x_2, x_4, \dots, x_{2k-2}\} \rangle) = \emptyset,$$

i.e.  $D \equiv K_{n/2, n/2}^* - \{e\}$ . Claim 6 is proved.  $\square$

**Claim 7.** If  $a(y, z) = 2$  and  $d(y) = n$ , then  $n$  is even and either  $D \equiv K_{n/2, n/2}^*$  or  $D \equiv K_{n/2, n/2}^* - \{e\}$ , where  $e$  is an arc of  $K_{n/2, n/2}^*$ .

**Proof.** For definite let  $x_m y \in D$  and  $a(y, x_1) = 0$ . Then  $\{y, x_1\}$  is a good pair. By Claim 3,  $a(y, x_2) \neq 0$ , and since  $d(y, C) = n-2$ , using Lemma 2(i) we obtain that  $yx_2 \in D$ . It is not difficult to see that  $m \geq 4$ , and by Claim 4,  $x_{m-1} y \notin D$ . From  $x_m y, yx_2 \in D$  and Claim 5 it follows that  $yx_3 \notin D$ . Now by Lemma 2(ii) we have  $d(y, C[x_3, x_m]) \leq n-4$ . Therefore  $x_2 y \in D$  and by Claim 4,  $a(x_3, y) = 0$  and hence,  $d(y, C[x_4, x_m]) = n-4$ . Again using Lemma 2(ii) we obtain that  $yx_4 \in D$  and hence by Claim 5,  $yx_5 \notin D$ . Therefore, by Lemma 2(ii),  $d(y, C[x_5, x_m]) \leq n-6$  and hence,  $x_4 y \in D$  and  $a(y, x_5) = 0$ . Continuing this process we finally conclude that  $n := 2k$  and

$$a(y, x_2) = a(y, x_4) = \dots = a(y, x_{2k-2}) = 2, \quad a(y, x_1) = a(y, x_3) = \dots = a(y, x_{2k-3}) = 0.$$

Observe that  $d(x_{2i-1}) \geq n-1$  by (\*) for every  $i \in [1, k-1]$ . Consider the cycle  $C_{n-2} := x_{2i} y x_{2i+2} \dots x_{2i}$  of length  $n-2$ ,  $i \in [1, k-1]$ . Note that the vertices  $x_{2i+1}$  and  $z$  are not on this cycle. We can assume that  $a(z, x_{2i+1}) = 2$  (otherwise, by Claim 6,  $D \equiv K_{n/2, n/2}^* - \{e\}$ ). Analogously to the proof of Claim 6, we get that

$$A(\langle \{y, x_1, x_3, \dots, x_{2k-3}\} \rangle) = A(\langle \{z, x_2, x_4, \dots, x_{2k-2}\} \rangle) = \emptyset.$$

Now using the condition (\*) and the fact that for every pair of distinct  $i, j \in \{1, 3, \dots, 2k-3\}$  ( $i, j \in \{2, 4, \dots, 2k-2\}$ ),  $\{x_i, x_j\}$  is a good pair, we conclude that either  $D \equiv K_{n/2, n/2}^*$  or  $D \equiv K_{n/2, n/2}^* - \{e\}$ . Claim 7 is proved.  $\square$

Let us now complete the proof of the theorem. By Claims 6 and 7 we can assume that for any cycle of length  $n-2$  in  $D$  if the vertices  $u$  and  $v$  are not on this cycle then  $\max\{d(u), d(v)\} \leq n-1$  and  $a(u, v) = 2$ .

W.l.o.g. assume that  $x_m y \in D$  and  $a(y, x_1) = 0$ , i.e.  $\{y, x_1\}$  is a good pair. Then  $d(y) = n-1$  and  $d(x_1) = n$  by our assumption and (\*). Then  $a(y, x_2) \neq 0$  by Claim 3. Let  $yx_2 \in D$ , then  $C_{n-2} := x_m y x_2 \dots x_m$  and  $d(x_1) = n$ , which contradicts to our assumption. Let now  $yx_2 \notin D$ . Then  $x_2 y \in D$  and since  $d^+(y) \geq 2$  and  $d(y, C[x_2, x_m]) = n-3$ , it is not difficult to see that for some  $j \in [2, m-2]$ ,  $x_j y, yx_{j+2} \in D$  and  $a(y, x_{j+1}) = 0$ . A similar argument applies for this case, we again obtain a contradiction. The proof of Theorem 1 is complete.  $\square$

The following example shows that the sharpness the minimum semi-degree condition in Theorem 1 would be best possible in the sense that for all  $n = k+2 \geq 6$  there is a strong digraph  $D$  on  $n$  vertices which has minimum semi-degree one and satisfies the condition (\*) of Theorem 1, but contain no cycle of length  $n-1$ . To see this, let  $D$  be a digraph with vertex set  $V(D) = \{y, z, x_1, x_2, \dots, x_k\}$ ; and let (for the convenience of the reader)  $N^-(y) = \{z, x_1, x_3, x_4, \dots, x_k\}$  and  $N^+(y) = \{z\}$ ;  $N^-(z) =$

$\{y, x_1, x_2, x_4, x_5, \dots, x_k\}$  and  $N^+(z) = \{y, x_4\}$ ;  $N^-(x_1) = \{x_k, x_2, x_3\}$  and  $N^+(x_1) = \{y, z, x_2, x_4\} \cup \{x_5, x_6, \dots, x_{k-1}\}$ ;  $N^-(x_2) = \{x_1, x_3\}$  and  $N^+(x_2) = \{z, x_1, x_3, x_4, \dots, x_k\}$ ;  $N^-(x_3) = \{x_2\}$  and  $N^+(x_3) = \{y, x_1, x_2, x_4, x_5, \dots, x_k\}$ ;  $N^-(x_4) = \{z, x_1, x_2, x_3\} \cup \{x_6, x_7, \dots, x_k\}$  and  $N^+(x_4) = \{y, z, x_1\}$  if  $k = 4$  and  $N^+(x_4) = \{y, z, x_5\}$  if  $k \geq 5$ ; if  $5 \leq i \leq k-1$ , then  $N^-(x_i) = \{x_1, x_2, x_3, x_{i-1}\} \cup \{x_{i+2}, x_{i+3}, \dots, x_k\}$  and  $N^+(x_i) = \{y, z, x_{i+1}\} \cup \{x_4, x_5, \dots, x_{i-2}\}$ ; finally if  $k \geq 5$ , then let  $N^-(x_k) = \{x_2, x_3, x_{k-1}\}$  and  $N^+(x_k) = \{y, z, x_1\} \cup \{x_4, x_5, \dots, x_{k-2}\}$ , where  $\{x_i, x_{i+1}, \dots, x_j\} = \emptyset$  if  $j \leq i-1$ .

Note that  $x_1x_2 \dots x_kx_1$  is a cycle of length  $k = n-2$ ,  $\langle \{x_1, x_2, \dots, x_k\} \rangle$  is a semicomplete digraph, the pairs of non-adjacent distinct vertices with a common in-neighbour are only  $\{y, x_2\}$  and  $\{z, x_3\}$ ,  $d(y) = d(x_3) = n-1$ ,  $d(z) = d(x_2) = n$  and  $d^+(y) = d^-(x_3) = 1$ . It is not difficult to check that  $D$  is strong, satisfies the condition (\*) of Theorem 1 and contain no cycle of length  $n-1$ .

Moreover the following example from [14] (also [1], p. 300) also shows that in Theorem 1 the minimum semi-degree condition ( $\geq 2$ ) cannot be replaced by one. For some  $m \leq n$  let  $D_{n,m}$  be the digraph with vertices  $V(D_{n,m}) = \{x_1, x_2, \dots, x_n\}$  and arcs  $A(D_{n,m}) = \{x_ix_j / i < j \text{ or } i = j+1\} \setminus \{x_ix_{i+m-1} / 1 \leq i \leq n-m+1\}$ .  $D_{n,m}$  is strong, has no cycle of length  $m$  and if  $m = n-1$ , then the pairs  $\{x_1, x_{n-1}\}$  and  $\{x_2, x_n\}$  are only the non-adjacent pairs with a common in-neighbour. It is easy to check that  $d^-(x_1) = d^+(x_n) = 1$ ,  $d(x_1) = d(x_n) = n-1$  and  $d(x_{n-1}) = d(x_2) = n$ .

**Theorem 2.** Let  $D$  be a strong digraph on  $n \geq 4$  vertices, which is not directed cycle of length  $n$ . Suppose that

$$\min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq n \quad (**)$$

for every pair of non-adjacent vertices  $\{x, y\}$  with a common out-neighbour or a common in-neighbour. Then either  $D$  contains a cycle of length  $n-1$  or  $n$  is even and  $D$  isomorphic to complete bipartite digraph  $K_{n/2, n/2}^*$ .

**Proof.** Suppose that  $D$  has no cycle of length  $n-1$  and  $C := x_1x_2 \dots x_mx_1$  is a longest non-hamiltonian cycle in  $D$ . Let  $R := V(D) - V(C)$ . Then  $3 \leq m \leq n-2$ , i.e.  $|R| \geq 2$ . In [2] (see [1] page 250), was shown that  $D$  has a  $C$ -bypass with three vertices. W.l.o.g. assume that  $B := x_1yx_{j+1}$  is a  $C$ -bypass and the gap  $j$  of  $B$  with respect to  $C$  is minimum among the gaps of all  $C$ -bypasses with three vertices. Clearly,  $j \geq 2$  and

$$A(y, \{x_2, x_3, \dots, x_j\}) = \emptyset. \quad (10)$$

Observe that  $\{y, x_2\}$  ( $\{y, x_j\}$ , respectively) is a pair of non-adjacent vertices with a common in-neighbour  $x_1$  (with a common out-neighbour  $x_{j+1}$ , respectively). Therefore for these pairs the condition (\*\*) of the theorem holds.

Let  $C'' := C[x_{j+1}, x_1]$  and  $C' := C[x_2, x_j]$ . By Lemmas 2, 3 and the maximality of  $C$  we have

$$d(y, C'') \leq |C''| + 1; \quad d^-(x_2, C'') + d^+(x_j, C'') \leq |C''| + 1. \quad (11)$$

**Case 1.**  $|R| \geq 3$ , i.e.  $m \leq n-3$ . Then

$$d^+(y, R) + d^-(x_2, R) \leq n-m-1 \quad \text{and} \quad d^+(x_j, R) + d^-(y, R) \leq n-m-1,$$

otherwise,  $D$  contains a long non-hamiltonian cycle than  $C$ . This along with (10), (11) and (\*\*) gives

$$2n \leq d^-(y) + d^+(x_j) + d^+(y) + d^-(x_2) = d^-(y, R) + d^+(x_j, R) + d^+(y, R) + d^-(x_2, R) + d(y, C'') +$$

$$d^+(x_j, C'') + d^-(x_2, C'') + d^-(x_2, C') + d^+(x_j, C') \leq 2(n-m-1) + 2|C''| + 2 + 2|C'| - 2 \leq 2n-2$$

since  $|C'| + |C''| = m$ , which is a contradiction.

**Case 2.**  $|R| = 2$ , i.e.  $m = n-2$ . Let  $R = \{y, z\}$ .

**Subcase 2.1.**  $j = 2$ , i.e.  $x_1y, yx_3 \in D$  and  $a(x_2, y) = 0$ . Then  $|C''| = n - 3$ . By the maximality of the cycle  $C$  and Lemma 1 we have  $d(y, C), d(z, C) \leq n - 2$ . From the condition (\*\*) of the theorem it follows that  $d(y) \geq n$  or  $d(x_2) \geq n$ . W.l.o.g. we assume that  $d(y) \geq n$ . Then

$$n \leq d(y) = a(y, z) + d(y, C) \leq 2 + d(y, C).$$

Since  $d(y, C) \leq n - 2$ , it follows that  $a(y, z) = 2$ ,  $d(y, C) = n - 2$  and  $d(y) = n$ . Similarly, we obtain that have  $d(x_2) = n$ ,  $d(x_2, C) = n - 2$  and  $a(x_2, z) = 2$  (by (\*\*) and (11)). It is easy to see that  $n \geq 6$ . Observe that  $yx_4 \notin D$  and  $x_my \notin D$ . Now using Lemma 2(iii) we obtain,  $d(y, C[x_4, x_m]) = n - 6$  and  $a(x_1, y) = a(y, x_3) = 2$ . If  $n = 6$ , then it is easy to check that  $D \equiv K_{3,3}^*$ . Assume that  $n \geq 7$ . If  $\{x_{k-1}, x_k\} \rightarrow y$  and  $a(y, x_{k+1}) = 0$  for some  $k \in [4, m - 1]$ , then by Lemma 2(ii),  $yx_{k+2} \in D$ . Then, since  $\{y, x_{k+1}\}$  is a pair of non-adjacent vertices with a common in-neighbour  $x_k$ ,  $d(y) = n$  and the vertex  $x_{k+1}$  has no partner on  $C[x_{k+2}, x_k]$  it follows that  $d(x_{k+1}) = n$  and  $a(x_{k+1}, z) = 2$ . Therefore  $C_{n-1} := x_{k+1}C[x_{k+2}, x_{k-1}]yzx_{k+1}$ , a contradiction. So we can assume that  $d^-(y, \{x_i, x_{i+1}\}) \leq 1$  for all  $i \in [1, m]$ . This together with  $x_3y \in D$  and  $yx_4 \notin D$  implies that  $a(x_4, y) = 0$ . Analogously above, we obtain that  $d^+(y, \{x_i, x_{i+1}\}) \leq 1$  for all  $i \in [1, m]$ . Now it is not difficult to see that  $n$  is even ( $n := 2k + 2$ ),  $a(y, x_i) = 2$  for all  $i \in \{1, 3, \dots, 2k - 1\}$  and  $A(y, \{x_2, x_4, \dots, x_{2k}\}) = \emptyset$ . Then  $\{y, x_{2j}\}$  is a pair of non-adjacent vertices with a common in-neighbour  $x_{2j-1}$  for all  $j \in [1, k]$ . Therefore  $d(x_{2j}) = n$  since  $d(y) = n$ , and  $a(z, x_{2j}) = 2$ ,  $A(z, \{x_1, x_3, \dots, x_{2k-1}\}) = \emptyset$  since  $x_{2j}$  cannot be inserted into  $C[x_{2j+1}, x_{2j-1}]$ . We finally conclude that either  $D$  contains a cycle of length  $n - 1$  or  $D \equiv K_{n/2, n/2}^*$  with partite sets  $\{z, x_1, x_3, \dots, x_{2k-1}\}$  and  $\{y, x_2, x_4, \dots, x_{2k}\}$ .

**Subcase 2.2.**  $j \geq 3$ . From (11) and (\*\*) it follows that

$$\min\{d(x_2), d(x_j)\} \geq 2n - d(y) = 2n - d(y, C'') - a(y, z) \geq n + j - a(y, z).$$

Therefore

$$d(x_2, C'') \geq n + j - 2(|C'| - 1) - d(z, \{y, x_2\}) \geq n - j + 4 - d(z, \{y, x_2\})$$

and similarly

$$d(x_j, C'') \geq n - j + 4 - d(z, \{y, x_j\}).$$

From this it follows that if  $j = 3$ , then  $d(z, \{y, x_2\}), d(z, \{y, x_2\}) \leq 3$  and  $d(x_2, C''), d(x_j, C'') \geq n - j + 1 = |C''| + 2$ . So, by Lemma 2(i) we have that  $x_2$  and  $x_3$  has a partner on  $C''$  and therefore,  $D$  contains a cycle of length  $n - 1$ , a contradiction. Now we can assume that  $j \geq 4$ . Note that  $d(y, C) \leq n - j$  by Lemma 2(i).

First assume that  $a(y, z) \leq 1$ . Then  $d(y) \leq n - j + 1$  and by (\*\*)

$$2n \leq d^+(y) + d^-(x_2) + d^-(y) + d^+(x_j) \leq n - j + 1 + d^-(x_2) + d^+(x_j)$$

and

$$d^-(x_2) + d^+(x_j) \geq n + j - 1. \tag{12}$$

This together with (11) implies that

$$n + j - 1 \leq d^-(x_2) + d^+(x_j) \leq n - j + 2(j - 2) + 2 = n + j - 2,$$

which is a contradiction.

Second assume that  $a(y, z) = 2$ . Then  $d(y) \leq n - j + 2$  and similarly (12) we obtain that  $d^-(x_2) + d^+(x_j) \geq n + j - 2$ . On the other hand using (11) it is easy to see that  $d^-(x_2) + d^+(x_j) \leq n + j - 2$ . Therefore  $d^-(x_2) + d^+(x_j) = n + j - 2$ ,  $zx_2, x_jz \in D$  and  $d(y, C'') = n - j \geq 3$ . From this it is not difficult to see that  $x_my \notin D$  and  $yx_{j+2} \notin D$ . Therefore  $n - j \geq 4$ , and by Lemma 2(iii),  $d(y, C[x_{j+2}, x_m]) \leq n - j - 4$ . Hence  $yx_1, x_{j+1}y \in D$  and  $d(y, C[x_{j+2}, x_m]) = n - j - 4$ . Now using Lemma 2 we obtain that  $x_{i-1}y, yx_{i+1} \in D$

and  $a(x_i, y) = 0$  for some  $i \in [j+2, m]$ , i.e. we have the considered Subcase 2.1. The theorem is proved.  $\square$

We believe Theorem 2 can be generalized to the following

**Conjecture.** Let  $D$  be a strong digraph on  $n \geq 4$  vertices. Suppose that  $\min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq n - 1$  and  $d(x) + d(y) \geq 2n - 1$  for every pair of non-adjacent vertices  $x, y$  with a common out-neighbour or a common in-neighbour (i.e., satisfies the conditions of Theorem C). Then  $D$  contains a cycle of length  $n - 1$  maybe except some digraphs which has a "simple" characterization.

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